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Gear Couplings

A discussion and mathematical analysis of the operation of gear couplings at angular misalignment; transmission of uniform motion, tooth separation, tooth load distribution, coupling load capacity, tooth bearing, and special tooth forms.

There are differences of opinion as to how flexible gear couplings act, whether they transmit true uniform motion at substantial angles, how many teeth are in contact, and what kind of crown should be applied.

We shall now try to establish the facts pertinent to this subject.

Gear couplings are usually arranged in pairs, each individual coupling comprising a sleeve with straight teeth on its inside, and a hub with teeth crowned to cooperate with the sleeve at the range of angles specified.

A sleeve, *S*, and hub, *H*, are shown in aligned position in a fragmentary cross section in Fig. 1 and in an axial section in Fig. 2.

Generally, the sleeve teeth have involute profiles, *inv*, rising from a base circle, *b*, as on conventional gears. Adjacent involute tooth surfaces have a constant distance p from each other anywhere, taken in the direction of the surface normal q. In consequence, the adjacent crowned tooth surfaces of the hub should also have a constant distance from each other in the direction of the surface normals to match the sleeve teeth. This requirement is no different on couplings than it is on gears. Teeth with unequal normal distance p could not be brought to match and take over load smoothly from one another.

As a result of this requirement, the tooth profiles of the hub, in planes g perpendicular to the hub axis, should change increasingly with increasing distance from the hub axis, at least when the coupling is designed for a substantial running angle. This will be further described.

Fig. 2 shows a conventional uniformly crowned hub. The hub looks like an excessively crowned gear. As on a gear, its shape is best defined by the shape of the rack teeth with which it can mesh and run so that its entire tooth sides get into contact.

This rack can be considered an extremely large, infinitely large, gear. A hob produces the straight profile of the involute rack in the midplane G. If now the hob is fed about an axis C as if turned about this axis, it will produce the same straight profile in all planes containing axis C and envelop the rack tooth shape. Each tooth surface of the rack contains straight profiles that intersect axis C at the same point and that have a constant inclination to axis C. In other words, the surface that would be produced on the rack is a conical surface with axis C.

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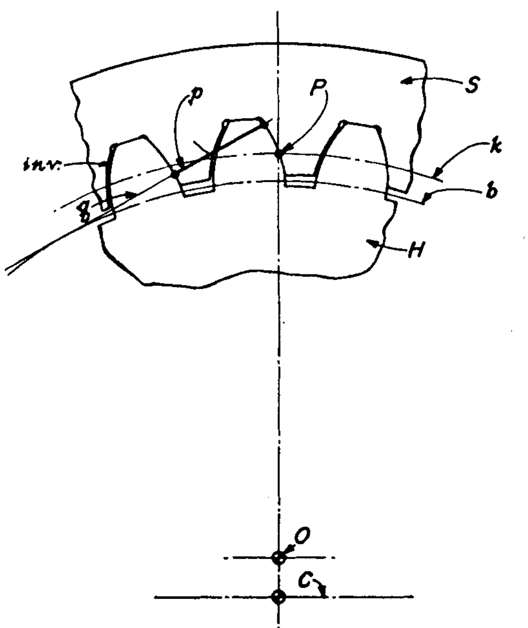

Fig. 1

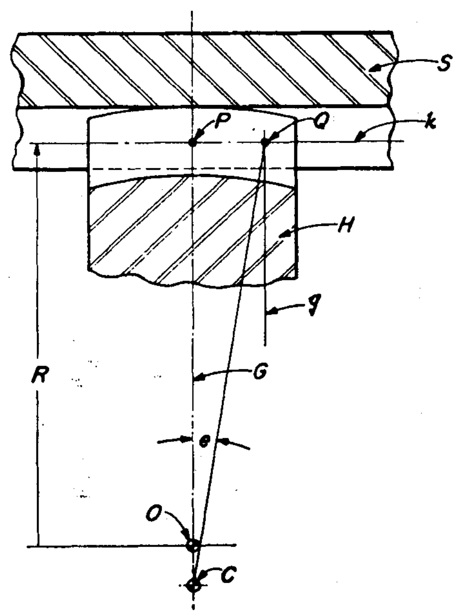

Fig. 2

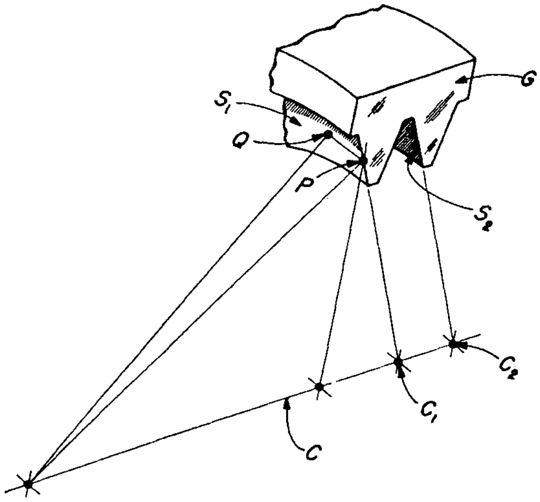

Fig. 3

In its feed motion about axis C, the rotating hob, so to say, represents a moving rack that is the counterpart of the rack or infinitely large gear produced by the hob. Fig. 3 shows part of the represented rack with axis C in perspective. The straight rack profiles show up in all axial sections, as in midplane G.

They come to a point or apex on axis C. The cone apex of side S_1 is at C_1, the one of side S_2 at C_2. Adjacent sides S_1, S_2 are identical conical surfaces merely displaced along the cone axis C. They have a constant distance from each other everywhere in the direction of the surface normals.

The tooth sides of the hub are produced in a generating motion whereby the rack represented by the hob rolls on the hub. This rolling motion is as if a cylindrical pitch surface k of the hub would roll on the pitch plane of the rack in contact therewith. The pitch surfaces intersect the tooth sides in curves called pitch lines. As the pitch surfaces roll on each other without sliding, the pitch lines of the hub are as if printed on its cylindrical pitch surface by the pitch lines of the rack. In development of the cylindrical pitch surface to a plane, the pitch lines of the hub are identical with the pitch lines of the rack.

At any intersection point Q of a rack pitch line with the contact line PQ of the pitch surfaces, the contacting tooth surfaces have the same direction. They have a common surface normal QC (Fig. 2), whose inclination to axis C is constant and equal to the pressure angle in the midplane G.

Normal QC remains normal to the conical rack tooth sides in its fixed position even as the rack moves along axis C. In each rack position, it intersects the rack-tooth side at a point of contact with the hub meshing therewith. It is a path of contact. As this is true for all points Q and their surface normals, these surface normals determine and define the surface of progressive contact. As in gearing, this surface can be used to establish the required tooth shape of the hub.

As the conical rack-tooth sides, such as S_1, S_2, have a constant normal distance from each other, the so-generated hub-tooth sides also have a constant normal distance from each other, as required for smooth power transmission.

At any point Q, the profile inclination or pressure angle is constant in plane QC (Fig. 2) that contains the cone axis C. Hence it is bound to be different in plane g that is parallel to midplane G and perpendicular to the hub axis. It can be shown that the profile inclination ϕ' or pressure angle in plane g is related to the pressure angle ϕ in midplane G and to angle $QCP = e$ as follows:

$$\tan \phi' = \tan \phi \cos e \qquad (1)$$

The term ϕ' is smaller than ϕ. It drops off first very slowly and then more rapidly with increasing angle e.

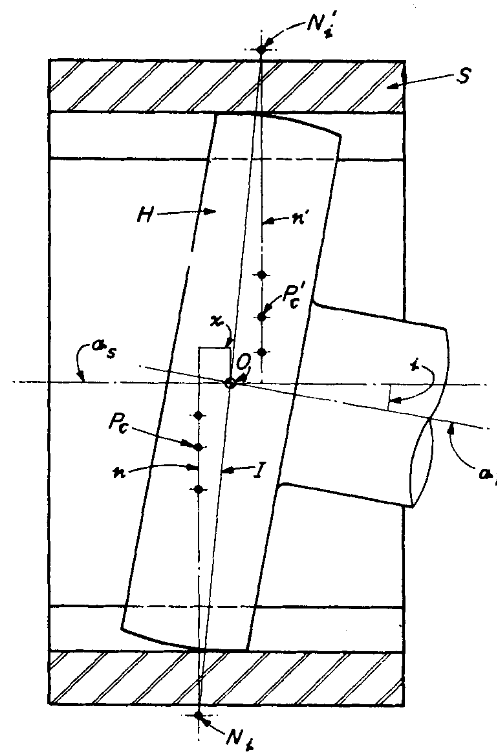

Fig. 4

Mesh

To describe the coupling action, we shall first look at the tooth contact without appreciable load and without any elastic deflection of the contacting tooth surfaces and teeth. The effect of elastic yielding will be introduced later. We shall also show that involute gear couplings can transmit true uniform motion, even at very large coupling angles.

Fig. 4 is an axial section showing a sleeve S and a hub H at an angle i. Here the tooth contact has shifted toward the tooth ends. Fig. 5 is an axial end view of the sleeve that has straight involute teeth with base circle b.

The uniform motion of the sleeve and hub is like the motion of a pair of bevel gears whose axes coincide with the sleeve and hub axes a_s, a_h that intersect at O. The sleeve and hub move as if two imagined conical pitch surfaces of the gear members roll on each other without sliding. These rolling surfaces contact along the instant axis I that bisects the angle between the coupling axes.

The instant axis I is very useful to show up the relative motion of one member with respect to the other. At any one instant, the hub moves briefly as if turned about the instant axis relatively to the sleeve. This defines the direction of relative motion of each point of the hub. The velocity of this motion depends on the turning velocity ω_i about the instant axis. It is known to be obtainable from the turning velocity ω of the sleeve and hub by geometric addition, as expressed in formula (2):

$$\omega_i = 2\omega \sin\left(\frac{i}{2}\right) \qquad (2)$$

The relative velocity of any point is the product of its distance from the instant axis and of ω_i. At a point of tooth contact, it defines the sliding velocity. It increases with increasing angle i.

As the instantaneous relative motion is a turning motion about the instant axis, the surface normal at any point of tooth contact is bound to intersect the instant axis I. With involute sleeve teeth, a surface normal fixed in space stays a surface normal at all turning angles of the sleeve. It becomes a path of contact. Uniform motion is transmitted. The contact point moving along

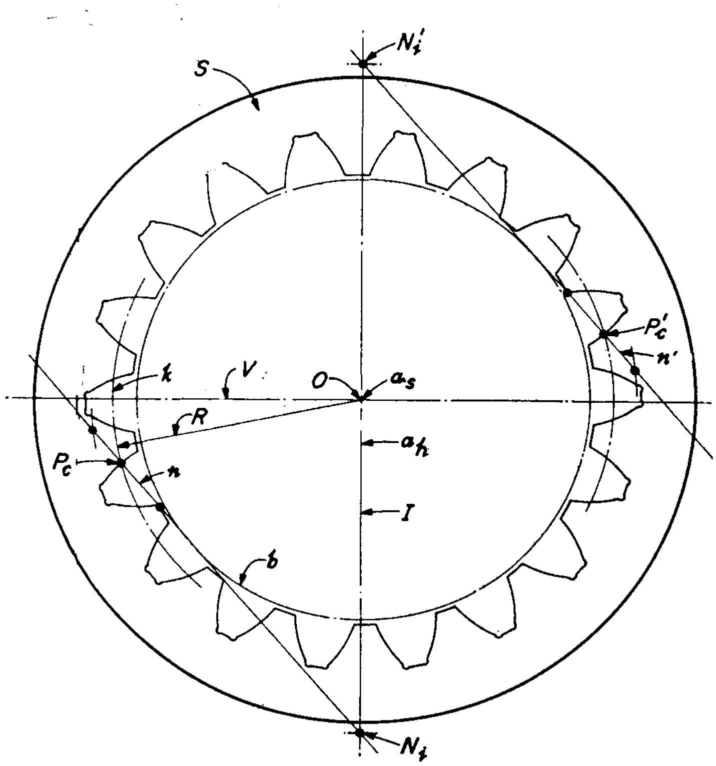

Fig. 5

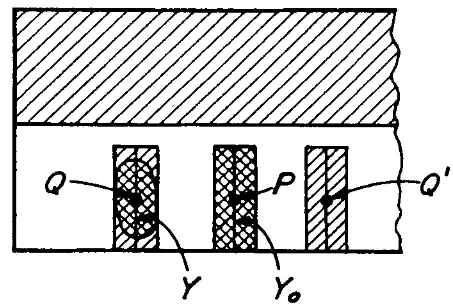

Fig. 6

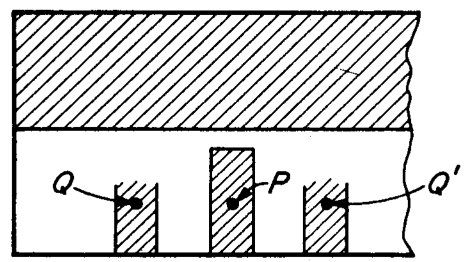

Fig. 7

its straight path describes a line on the tooth sides as the coupling turns. This line coincides with an involute profile on the sleeve teeth.

A path of contact n lies in a plane perpendicular to the sleeve axis a_s and when extended is tangent to base circle b (Fig. 5). It also intersects the instant axis I, at N_i.

The crown of the hub teeth can be determined so as to place the path of contact at a desired axial distance x from the intersection O of the axes, at the design misalignment angle i. The foregoing requirement defines the location of the contact point P_c at the pitch circle k. P_c generally does not lie directly on the vertical V through O, but close enough to it that its distance from the instant axis I does not differ much from pitch radius R. For this reason, the sliding velocity v_s at mean contact point P_c can be put down as

$$v_s = \frac{i^\circ}{60} v \text{ (approx)} \tag{3}$$

where v is the peripheral velocity $R \cdot \omega$, and i° is the coupling angle i measured in degrees.

Each of the two sides of the teeth has two diametrically opposite paths of contact. One is along normal n that intersects the instant axis I at N_i. The other is along normal n' that intersects instant axis I at N_i' on the opposite side from O. The contact normals n, n' intersect the cylindrical inside surface of the sleeve teeth and a spherical outer surface of the hub teeth. The path of contact is between the two intersection points. Its length determines the duration of contact. If it were exactly equal to the normal distance p (Fig. 1) of adjacent tooth sides, then each tooth starts contact when the preceding tooth leaves off. As on gears, profile overlap is desired, a length of $1.2p$ to $1.6p$ or more. This length and the duration of contact depend on the tooth depth and on the pressure angle or profile inclination.

After passing through the contact, a tooth separates from its mate, to contact it again at a different spot after about half a turn. The maximum separation attained depends on the coupling angle i.

Tooth Bearing

Fig. 6 is a fragmentary axial section of a sleeve, wherein its involute tooth profiles appear as straight lines. In the aligned position of the sleeve and hub, the tooth contact is along profile y_0 at zero load, all the teeth contacting simultaneously. On both sides of line y_0, the contacting tooth surfaces gradually separate from each other at a rate depending on the amount of crowning. They separate at first very slowly. The cross-hatched area around point P has a separation within a fixed, very small amount z', such as 0.001 in. Such a separation might be overcome by elastic deflection under heavy load. The area then becomes a tooth bearing area.

At a coupling angle i, the contact has shifted away from central position to two mesh zones. A tooth contacts only at one point at a time, at zero load, at a point such as Q in one turning position; and after half a turn at point Q'. The cross-hatched elliptical or oval area has a separation within a given small amount z'. It is smaller than the cross-hatched area around point P. As the coupling turns, the contact point moves along profile y. The rectangular area around Q or Q' is within a separation z' of getting into tooth contact at zero load. Under load, it may become the area swept by tooth contact.

With the conventional uniform crowning the width of these areas around points P and Q, Q' is approximately equal. When the coupling runs at an angle, however, there are fewer teeth in contact, only two at times at the maximum design angle, and these fewer teeth have less intimate contact. Moreover, sliding increases with increasing angle i. In consequence, the sustained load capacity at the design angle is only a small fraction of the capacity of the coupling in alignment or near-alignment.

Fig. 7 shows the kind of tooth bearing obtained at a substantial angle i when the profile inclination of the hub teeth is constant in planes g, Fig. 2, perpendicular to the hub axis rather than being constant in planes QC containing axis C. The profile inclination is then too large in planes g, so that the tooth bearing is displaced toward the top of the sleeve teeth when the coupling runs at the design angle. This affects the smoothness of the transmitted motion and causes early wear.

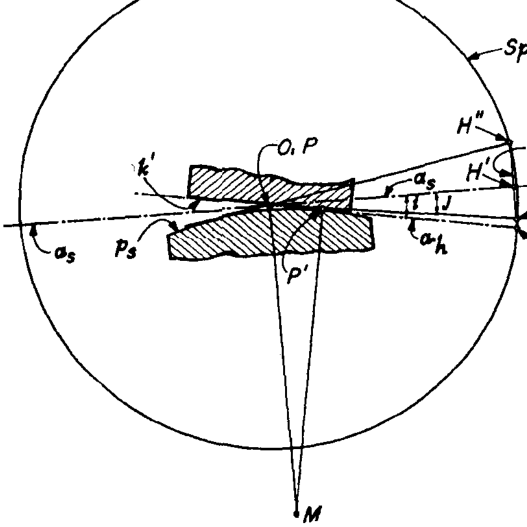

Fig. 8

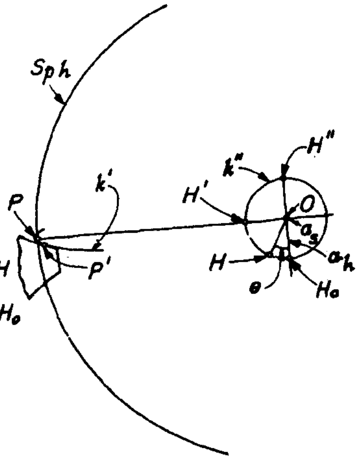

Fig. 9

Contact Cycles and Backlash

Let us look at the relative motion of the hub with respect to a sleeve maintained stationary. Instead of turning both the sleeve and hub on their axes, an opposite turning motion about the sleeve axis is added to the system comprising sleeve and hub, so that the sleeve turning motion is cancelled out and the sleeve stands still. The hub axis then describes a conical surface about the sleeve axis. Its apex is at the intersection point O of the axes.

We shall first consider the case where the crowning axis C intersects the hub axis, at O, and look at a spherical surface Sph centered at O and passing through mean point P of the hub. Fig. 8 is a radial view taken in direction PO. Fig. 9 is a side view taken in the direction of the sleeve axis a_s. The spherical surface Sph intersects the hub-tooth surface passing through P substantially in a circle (k') centered at O. In projection, Fig. 8, it appears as a straight line that coincides with the hub axis a_h. The same sphere Sph intersects the contacting tooth surface of the sleeve in a curve p_s whose mean curvature radius in projection, Fig. 8, can be shown to amount to R ctn ϕ on curves having only a small distance z_0 from O.

The circle and curve p_s contact or nearly contact at point P'. In the relative motion, the hub axis describes a conical surface about sleeve axis a_s, whereby a point H_0 of the hub axis describes a circle k''. At a turning angle θ, point H_0 reaches a position H. And at turning angle θ = 90 deg and θ = 180 deg, it reaches positions H' and H'', respectively. In the view in Fig. 8, the projected hub axis OH appears inclined at an angle j to sleeve axis a_s. Tan j can be readily computed as

$$\tan j = \tan i \cos \theta$$

At a turning angle of 90 deg, when H_0 is at H', the circle k' of the hub-tooth side again appears projected as a straight line in Fig. 8, a line coinciding with the projected hub axis OH'. And at a turning angle of θ = 180 deg, it appears projected as a straight line OH''. It appears in Fig. 8 as if swinging about radial line PO between end positions OH_0 and OH''.

We shall now compute the distances z between circle k' and curve p_s as if circle k' would swing about radial line PO whereby its plane always contains the hub axis. Although this assumption is not exactly fulfilled, it provides a close enough result at the moderate angles i now considered.

The curvature center M of projected curve p_s lies in the plane passing through P at right angles to the sleeve axis a_s, at a distance $\frac{R \text{ ctn } \phi}{\cos i}$ from P.

The distance of curvature center from projected circle k' is found to amount to $\frac{R \text{ ctn } \phi}{\cos i} \cdot \cos j$ and the distance z of circle k' from curve p_s is

$$z = R \text{ ctn } \phi \left(\frac{\cos j}{\cos i} - 1 \right)$$

In the more general cases where the distance R_c of mean point P from the crowning axis C-C (Fig. 2) differs from R, a sphere with radius R_c is considered. The foregoing formula for z applies also when R_c is substituted for R.

At small angles i, as in common use, the formula can be transformed into

$$z = {}^1/_2 R_c \text{ ctn } \phi \tan^2 i \sin^2 \theta$$

The maximum separation z_0 is attained when θ is 90 deg where $\sin \theta = 1$. Hence

$$z_0 = {}^1/_2 R_c \text{ ctn } \phi \tan^2 i \qquad (4)$$

$$z = z_0 \sin^2 \theta \qquad (5)$$

The coupling runs at minimum backlash at the maximum angle i. The backlash is increased by Δb when the coupling is set in alignment, whereby the separation z_0 is added on each side:

$$\Delta b = 2z_0 \qquad (6)$$

The foregoing figures apply to uniformly crowned teeth.

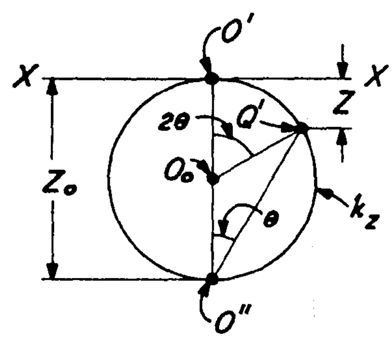

Fig. 10

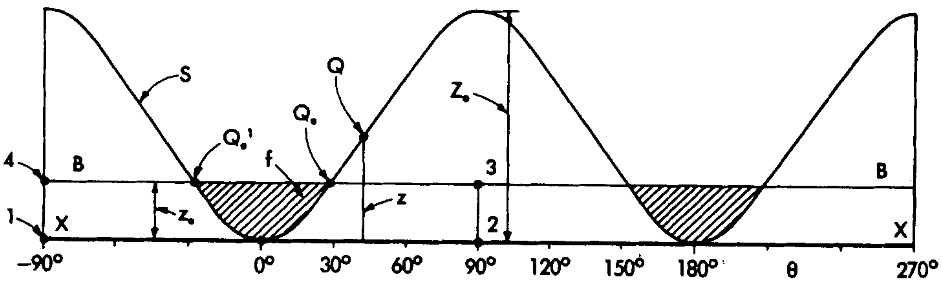

Fig. 11

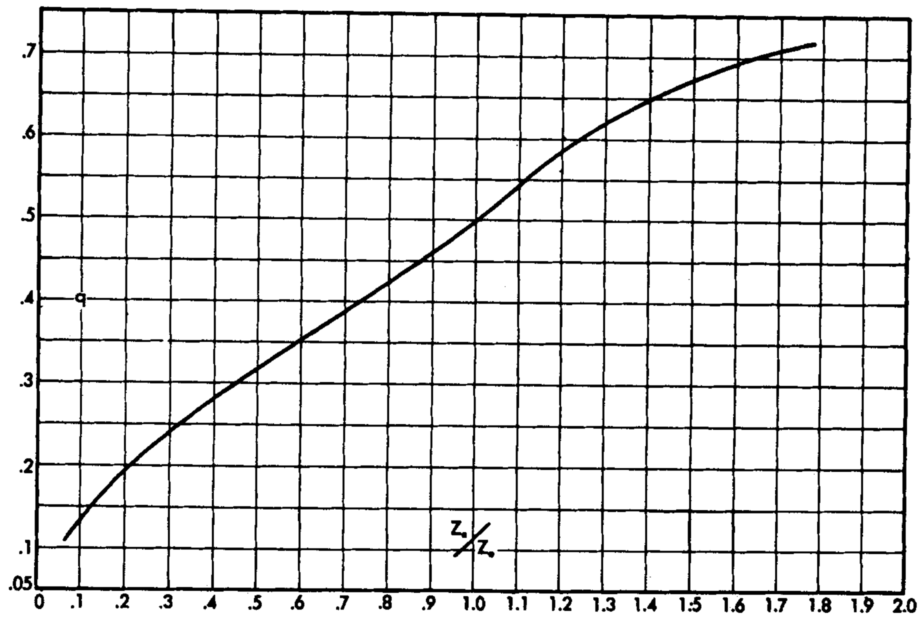

Fig. 12

The variation of separation z with the turning angle θ is directly shown in diagram, Fig. 10. It shows a circle k_s with center O_0 and diameter $z_0 = O'O''$. The term z shows up as the distance of any point Q' of the circle from the straight-line element X-X tangent to the circle at O'. Point Q' corresponds to a turning angle $\theta = Q'O''O'$. It can also be obtained by plotting an angle $2\theta'$ from center O_0. Separation z is seen to vary harmonically with the double turning angle 2θ.

Example: With $\phi = 20$ deg, $i = 1$ deg and $R_c = 2^1/_2$ in.; $z_0 = 0.0010$ in. from (4).

This is a quantity small enough that it compares with the elastic deflection of the teeth under load. Under load, then, more teeth get into simultaneous contact, especially at small angles i.

We shall now determine the number of teeth in contact under load.

Load Distribution

The number of teeth that carry the load depends on the angle i between the coupling axes, on the tooth design, and also on the load.

In operation at an angle at very small loads, each tooth gets into contact, separates, and contacts again after half a turn of the coupling. We have given the maximum tooth separation z_0 in formula (4) for no appreciable load and uniformly crowned teeth. The separation z at any turning angle θ from contact position is defined in formula (5).

To estimate the number of teeth in contact, we consider average conditions, without such irregularities in contact pattern as may occur when a new tooth gets into contact or a tooth gets out of contact. In the case considered here, moreover, the individual tooth load is in direct proportion to the elastic tooth deflection, the added deflection of the sleeve tooth and hub tooth, both surface deflection and bending. This proportionality is at least approximately fulfilled.

Those teeth are in contact whose separation z is smaller than the deflection z_e of the teeth that carry the largest individual load P_i, where

$$P_i = Cz_e$$

The proportionality factor C depends on the material used and on the tooth design. Its computation is involved and omitted here. It can also be determined reliably by test. In the test, all but two diametrically opposite teeth of the test hub are removed. Contact with the sleeve teeth is established at zero coupling angle. Then torque is applied which results in a slight relative turning displacement. The displacement is measured close to the contacting teeth. C is the proportion of the tooth load applied to the displacement at the pitch point.